Design & Analysis of Algorithms

Dynamic Programming

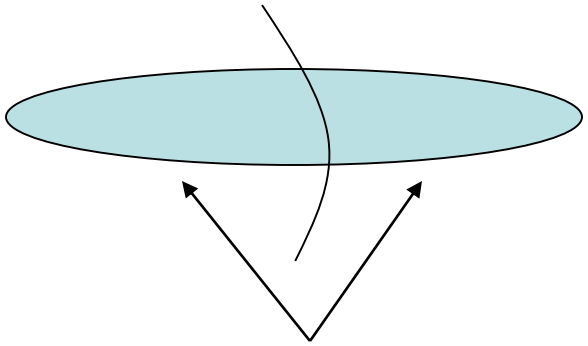
Dynamic Programming

- An algorithm design technique (like divide and conquer)
- Divide and conquer
  - Partition the problem into independent subproblems
  - Solve the subproblems recursively
  - Combine the solutions to solve the original problem

DP - Two key ingredients

- Two key ingredients for an optimization problem to be suitable for a dynamic-programming solution:

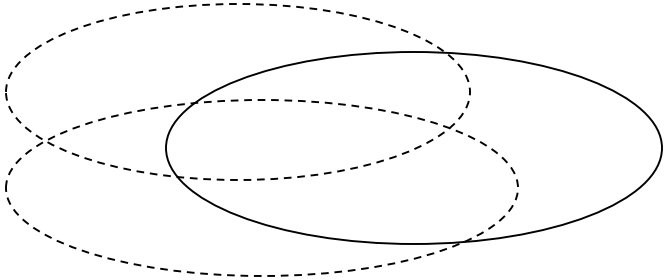
1. optimal substructures



Each substructure is optimal.

(Principle of optimality)

2. overlapping subproblems



Subproblems are dependent.

(otherwise, a divide-and-conquer approach is the choice.)

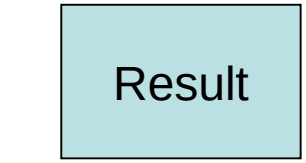
Three basic components

- The development of a dynamic-programming algorithm has three basic components:
  - The recurrence relation (for defining the value of an optimal solution);
  - The tabular computation (for computing the value of an optimal solution);
  - The traceback (for delivering an optimal solution).

Tabular computation

- The tabular computation can avoid recompuation.

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
0	1	1	2	3	5	8	13	21	34	55



Dynamic Programming Algorithm

1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution in a bottom-up fashion
4. Construct an optimal solution from computed information

The Knapsack Problem

- **The 0-1 knapsack problem**
  - A thief robbing a store finds n items: the i-th item is worth v_i dollars and weights w_i pounds (v_i, w_i integers)
  - The thief can only carry W pounds in his knapsack
  - Items must be taken entirely or left behind
  - Which items should the thief take to maximize the value of his load?
- **The fractional knapsack problem**
  - Similar to above
  - The thief can take fractions of items

The 0-1 Knapsack Problem

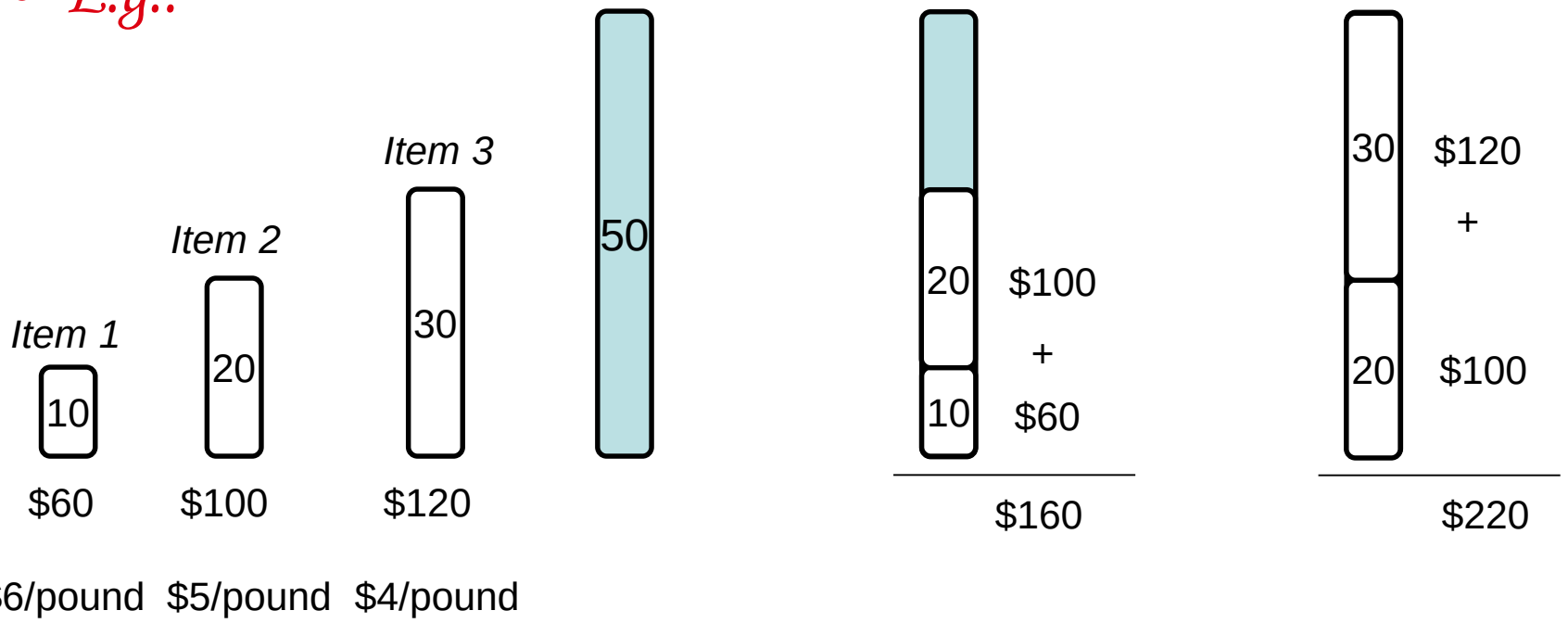
- Thief has a knapsack of capacity W
- There are n items: for i-th item value v_i and weight w_i
- Goal:
  - find x_i such that for all $x_i = \{0, 1\}$, $i = 1, 2, .., n$

    $\sum w_i x_i \leq W$ and

    $\sum x_i v_i$ is maximum

0-1 Knapsack - Greedy Strategy

- *E.g.:*



- None of the solutions involving the greedy choice (item 1) leads to an optimal solution
  - The greedy choice property does not hold

0-1 Knapsack - Dynamic Programming

- $P(i, w)$ – the maximum profit that can be obtained from items 1 to i, if the knapsack has size w
- Case 1: thief takes item i

$$P(i, w) = v_i + P(i - 1, w - w_i)$$

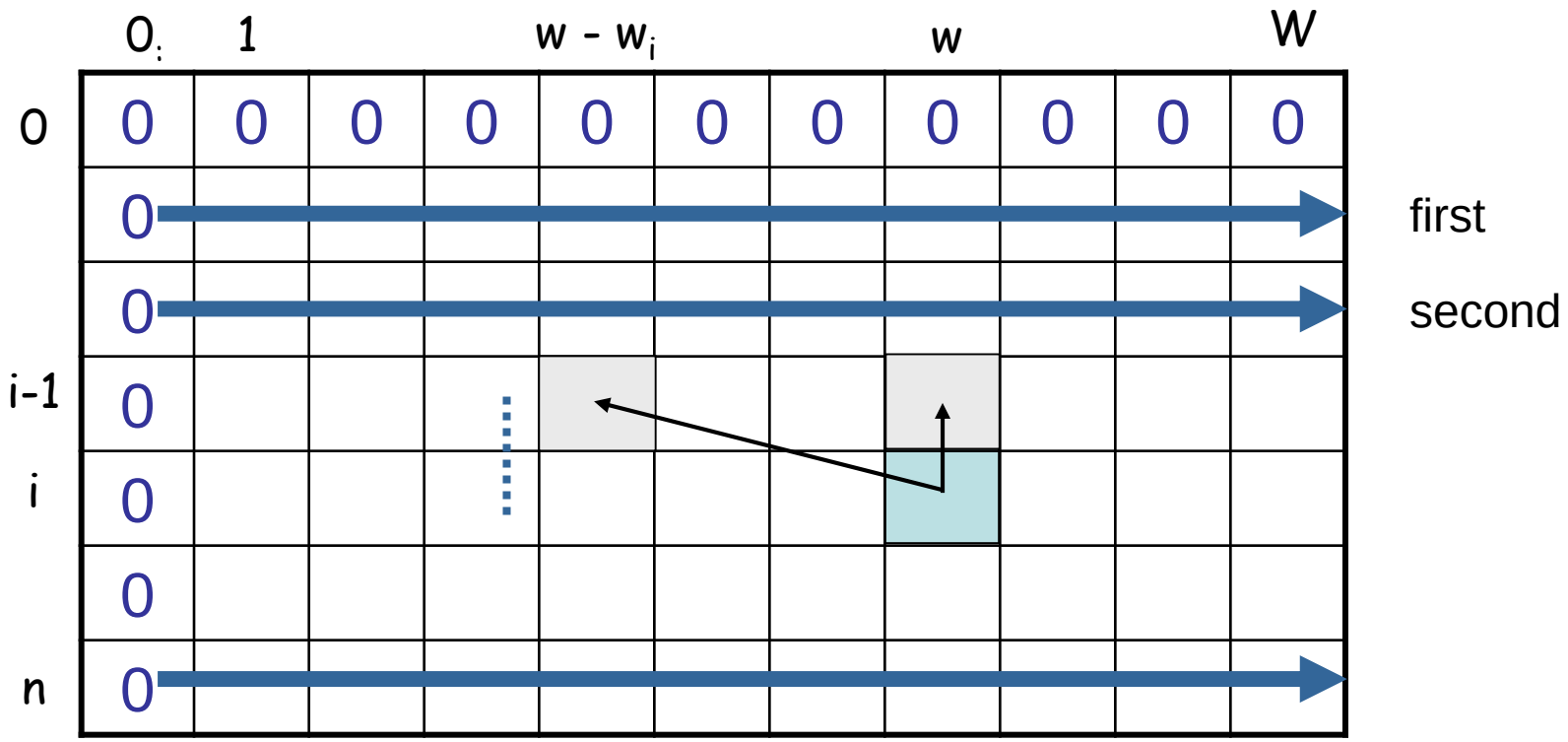
- Case 2: thief does not take item i

$$P(i, w) = P(i - 1, w)$$

0-1 Knapsack - Dynamic Programming

Item i was taken Item i was not taken

$$P(i, w) = \max \{v_i + P(i - 1, w-w_i), P(i - 1, w) \}$$



Example:

$P(i, w) = \max \{v_i + P(i - 1, w-w_i), P(i - 1, w) \}$

W = 5

Item	Weight	Value
1	2	12
2	1	10
3	3	20
4	2	15

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	37

P(1, 1) = P(0, 1) = 0

P(1, 2) = max{12+0, 0} = 12

P(1, 3) = max{12+0, 0} = 12

P(1, 4) = max{12+0, 0} = 12

P(1, 5) = max{12+0, 0} = 12

P(2, 1)= max{10+0, 0} = 10

P(2, 2)= max{10+0, 12} = 12

P(2, 3)= max{10+12, 12} = 22

P(2, 4)= max{10+12, 12} = 22

P(2, 5)= max{10+12, 12} = 22

P(3, 1)= P(2,1) = 10

P(3, 2)= P(2,2) = 12

P(3, 3)= max{20+0, 22}=22

P(3, 4)= max{20+10,22}=30

P(3, 5)= max{20+12,22}=32

P(4, 1)= P(3,1) = 10

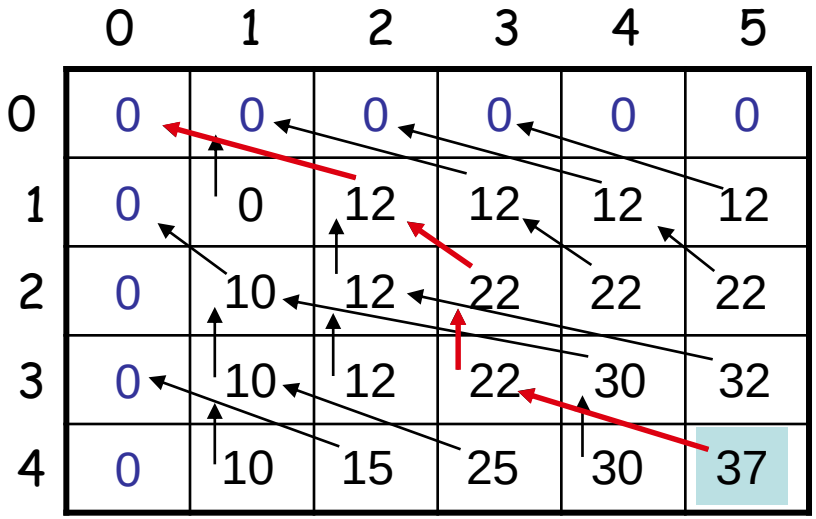
P(4, 2)= max{15+0, 12} = 15

P(4, 3)= max{15+10, 22}=25

P(4, 4)= max{15+12, 30}=30

P(4, 5)= max{15+22, 32}=37

Reconstructing the Optimal Solution



	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	12	12	12	12
2	0	10	12	22	22	22
3	0	10	12	22	30	32
4	0	10	15	25	30	37

- Item 4
- Item 2
- Item 1

- Start at $P(n, W)$
- When you go left-up $\Rightarrow$ item i has been taken
- When you go straight up $\Rightarrow$ item i has not been taken